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Skin friction and heat transfer on a continuous flat surface moving in a parallel free stream

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NOMENCLATURE

$\bar{b}$	plate width
C_f	local skin-friction coefficient
C_D	drag coefficient
F	dimensionless transformed stream function
G	dimensionless transformed temperature
$\bar{h}_x$	local heat transfer coefficient
$\bar{k}$	fluid thermal conductivity
$\bar{L}$	characteristic length
T	dimensionless temperature
$\bar{T}_w$	wall temperature
$\bar{T}_\infty$	free stream temperature
u	velocity component along the flat plate
u_w	normalized velocity of the continuous surface
u_∞	normalized free stream velocity
$\bar{u}_r$	reference velocity, equation (1)
v	velocity component normal to the flat plate
x	dimensionless coordinate along the flat plate
y	dimensionless coordinate normal to the plate
Nu	Nusselt number, $\bar{h}_x\bar{L}/\bar{k}$
Pr	Prandtl number, $\bar{\mu}\bar{c}_p/\bar{k}$
Re	Reynolds number, $\bar{u}_r\bar{L}/\bar{\nu}$.

Greek symbols

δ	boundary-layer thickness
ε	$Re^{-0.5}$
η	dimensionless similarity coordinate
$\bar{\mu}$	dynamic viscosity
$\bar{\nu}$	kinematic viscosity
$\bar{\rho}$	fluid mass density
$\bar{\tau}_w$	wall shear stress
ψ	dimensionless stream function.

1. INTRODUCTION

IN RECENT years consideration has been given to a somewhat novel type of flow situation that has been designated as the boundary layer on a continuous moving surface. The essential features of such a flow are illustrated in Fig. 1. A continuous plane sheet issues from a slit with a constant surface speed u_w in a parallel free stream u_∞.

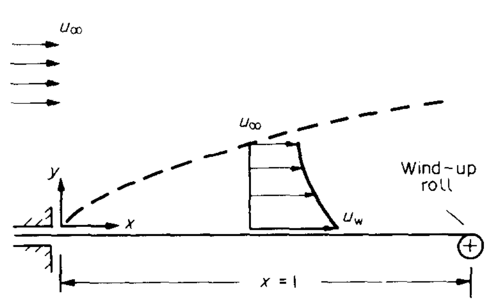

FIG. 1. Schematic representation of the boundary layer on a continuous moving surface.

Examples of practical applications of a continuous flat surface are: (i) the aerodynamic extrusion of plastic sheets; (ii) the cooling of an infinite metallic plate in a cooling bath; (iii) the boundary layer along material handling conveyers; and (iv) the boundary layer along a liquid film in condensation processes.

The problem of flow past a continuous moving flat surface (CF-problem) such as a flat belt, conveyer or an extruded sheet differs from the classical Blasius [1] problem of flow past a flat plate (BF-problem). In the CF-problem the flow field is steady for a coordinate system fixed in space, while in the BF-problem the flow field is steady for a coordinate system moving with the plate. Along the continuous moving surface and for a sufficiently large Reynolds number a boundary layer is developed whose thickness, in contrast to the classical BF-problem, increases in the direction of motion of the continuous surface. The continuous moving flat surface sucks the ambient fluid and pumps it again in the downstream direction. Consequently the transverse velocity component in the boundary layer is directed towards the plate when $u_w > u_\infty$. Such a pumping action of the continuous moving flat surface exists also when $u_w < u_\infty$ and thus reduces the displacement effect induced by the plate due to the formation of boundary layer. The change in direction of the transverse velocity component results in larger skin-friction and heat-transfer coefficients for $u_w > u_\infty$ than for $u_w < u_\infty$ for the same velocity difference $|u_w - u_\infty|$, Reynolds and Prandtl number. Thus the two CF- and BF-problems are physically different and can not be mathematically transformed into one another. The fact that the CF-problem cannot be transformed into the BF-problem indicates that we cannot regard the velocity difference $|u_w - u_\infty|$ as a relative velocity in the sense of Galilei.

The solution of the BF-problem depends on the relative velocity between the flat plate and the free stream, while the solution of the CF-problem depends not only on the velocity difference $|u_w - u_\infty|$ but also on the velocity ratio u_w/u_∞.

The boundary layer on a continuous semi-infinite sheet moving steadily through an otherwise quiescent fluid environment was first studied theoretically by Sakiadis [2]. The boundary-layer solution of Sakiadis resulted in a skin-friction coefficient along the continuous moving flat surface about 30% higher than that of Blasius for the flow past a stationary flat plate. Later an experimental and theoretical treatment was made for the CF-problem by Tsou *et al.* [3], who also determined heat transfer rates for certain values of the Prandtl number. In the previous literature the fluid and thermal boundary layers along a continuous surface moving through an otherwise quiescent fluid were analysed by solving the boundary-layer equations and the effect of an accompanying parallel free stream was not considered. It will be shown that Blasius [1] and Sakiadis [2] solutions are two special cases of the general problem using a method of normalization which shows clearly the effects of Reynolds number, velocity difference and velocity ratio. Using two different similarity transformations the boundary-layer equations are reduced to two different forms of a two-point nonlinear boundary-value problem giving nearly the same results by numerical integration using the Runge–Kutta technique.

Navier–Stokes solutions for the general CF-problem were first numerically developed by Abdelhafez [4]. In this paper we consider only the case of a continuous flat surface moving with steady surface speed u_w in a parallel free stream u_∞ in the same direction of motion of the continuous flat surface.

2. BOUNDARY-LAYER ANALYSIS

The flow field in the CF-problem, Fig. 1, is steady for a Cartesian coordinate (x, y) system fixed in space. Distances are normalized by the characteristic length $\bar{L}$ between the slit and the wind up roll or the portion of this length along which the flow is laminar (the overbars indicate dimensional quantities). Velocities are normalized by the largest velocity in the problem as a reference velocity $\bar{u}_r$:

$$\bar{u}_r = \begin{cases} \bar{u}_w & \text{if} \quad \bar{u}_w > \bar{u}_\infty \\ \bar{u}_\infty & \text{if} \quad \bar{u}_w < \bar{u}_\infty. \end{cases} \tag{1}$$

The velocity components corresponding to the x and y directions are respectively denoted by u and v. Defining the Reynolds number as:

$$Re = \bar{u}_r \bar{L}/\bar{v} = O(\varepsilon^{-2}), \quad \varepsilon \ll 1, \tag{2}$$

and introducing the dimensionless variables:

$$x = \bar{x}/\bar{L} = O(1), \quad y = \bar{y}/\bar{L} = O(\varepsilon), \quad u = \bar{u}/\bar{u}_r = O(1), \tag{3}$$
$$v = \bar{v}/\bar{u}_r = O(\varepsilon), \quad T = (\bar{T}-\bar{T}_\infty)/(\bar{T}_w - \bar{T}_\infty) = O(1),$$

result in the dimensionless boundary-layer equations (the subscripts denote partial differentiation):

$$u_x + v_y = 0, \tag{4}$$

$$uu_x + vu_y = u_{yy}/Re, \tag{5}$$

$$uT_x + vT_y = T_{yy}/(Re\ Pr), \tag{6}$$

with the boundary conditions:

$$y = 0: \quad u = u_w, \quad v = 0, \quad T = 1,$$
$$y = \delta: \quad u = u_\infty, \quad T = 0. \tag{7}$$

Introducing a dimensionless stream function ψ defined by:

$$\psi(x, y) = \bar{\psi}/(\bar{u}_r \bar{L}) = O(\varepsilon), \quad u = \psi_y, \quad v = -\psi_x, \tag{8}$$

eliminates the continuity equation (4). Employing the similarity dimensionless coordinate:

$$\eta(x, y) = y(Re/x)^{0.5} = O(1), \tag{9}$$

and the transformed dimensionless stream function $F(\eta)$ defined as;

$$F(\eta) = \psi(x, y)(Re/x)^{0.5}. \tag{10}$$

the boundary-layer momentum and mass conservation equations (5) and (4) may be reduced to the single ordinary differential equation:

$$2F''' + FF'' = 0 \tag{11}$$

where the primes denote differentiation with respect to η, the velocity components are given by:

$$u = F', \tag{12}$$

$$v = (\eta F' - F)/|2(xRe)^{0.5}|. \tag{13}$$

The boundary conditions (7) become:

$$F(0) = 0, \quad F'(0) = u_w, \quad F'(\eta_\delta) = u_\infty. \tag{14}$$

Equation (11) is immediately recognizable as the Blasius [1] equation but with the boundary conditions (14) which reduce to those of the classical Blasius problem for the case when $u_w = 0$, $u_\infty = 1$ and reduce to those of Sakiadis [2] for the case when $u_w = 1$, $u_\infty = 0$. Trying to use the same boundary conditions on F as those of Blasius results into two different differential equations for F as shown in the Appendix. This precludes the transformation of the continuous flat plate problem into the classical Blasius problem. The wall shear stress $\bar{\tau}_w$ may be represented by a dimensionless skin-friction coefficient as:

$$C_f = \bar{\tau}_w/\bar{\rho}\bar{u}_r^2 = F''(0)/(xRe)^{0.5}. \tag{15}$$

The total drag $\bar{F}_d$ acting on both sides of the plate is given by the drag coefficient C_D defined by:

$$C_D = \bar{F}_d/(2\bar{b}\bar{L}\bar{\rho}\bar{u}_r^2) = 4F''(0)/(Re)^{0.5}. \tag{16}$$

Consideration may now be given to the heat transfer problem. We consider here the case of uniform wall temperature T_w. The dimensionless temperature in terms of the similarity coordinate η is defined by:

$$G(\eta) = T(x, y) = O(1). \tag{17}$$

The energy equation (6) may be reduced to:

$$G'' + \tfrac{1}{2}Pr \cdot F \cdot G' = 0, \tag{18}$$

with the boundary conditions:

$$G(0) = 1, \quad G(\eta_{\delta_{th}}) = 0. \tag{19}$$

The local heat transfer coefficient along the wall, $\bar{h}_x$, may be expressed in a dimensionless form in terms of the Nusselt number Nu_x defined as follows:

$$Nu_x = \bar{h}_x \bar{L}/\bar{k} = -(\partial T/\partial y)_{y=0} = -G'(0)(Re/x)^{0.5}. \tag{20}$$

3. NAVIER–STOKES ANALYSIS

The full time-dependent Navier–Stokes equations for the problem of a continuous flat surface moving in a parallel free stream have been numerically integrated using finite differences [4]. Different space-centered finite difference schemes using explicit point-relaxation as well as implicit line-relaxation alternating direction methods have been developed and used. A fourth-order accurate compact implicit Hermetian technique [5] has also been developed and used.

4. RESULTS

Figures 2 and 3 show the results of similarity solution plotted as the skin friction and heat-transfer coefficients against the normalized velocity difference $|u_w - u_\infty|$. Using this method of normalization the solutions for all possible combinations of u_w and u_∞ lie on two curves. The upper curve refers to the case where the velocity of the plate is larger than the free stream velocity while the lower curve shows the results for the opposite case. The curves are independent of the magnitudes of velocities whose effect is contained in Re. The points B and S on these curves represent the Blasius and Sakiadis solutions respectively. Point S in Fig. 3 agrees with the solution of ref. [3].

The C_f value for a given value of Re depends not only on the normalized velocity difference $|u_\infty - u_w|$, but also on which moves faster, the plate or the free stream. The skin-friction coefficient is larger for $u_w > u_\infty$ than for $u_w < u_\infty$.

Consider e.g. the two cases: ($\bar{u}_w = 100$ m s^{-1}, $\bar{u}_\infty = 20$ m s^{-1}) and ($\bar{u}_w = 20$ m s^{-1}, $\bar{u}_\infty = 100$ m s^{-1}). The value of Re is the same in both cases and we have the same normalized velocity difference $|u_\infty - u_w| = 0.8$. However in the first case we have $u_w > u_\infty$ giving $C_f(xRe)^{0.5} = 0.38$ from the upper curve in Fig. 2 while for the second case we have $u_w < u_\infty$ giving $C_f(xRe)^{0.5} = 0.34$ from the lower curve in Fig. 2.

Figure 4 shows the local skin-friction coefficient along the flat plate for cases when the plate moves faster than the free stream obtained by Navier–Stokes and boundary-layer theory. The Navier–Stokes solution gives higher C_f values

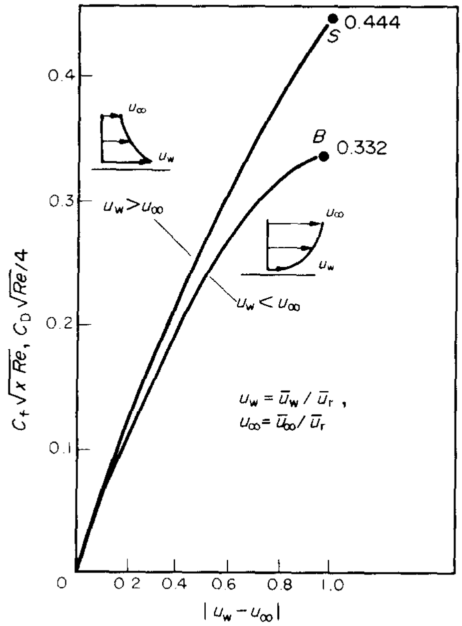

FIG. 2. Skin friction coefficient.

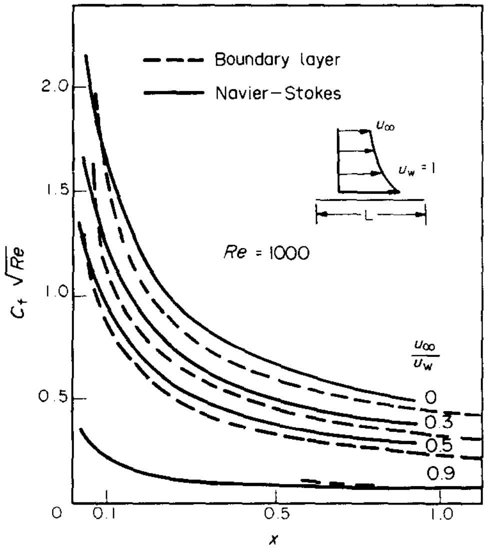

FIG. 4. Boundary-layer and Navier–Stokes solutions.

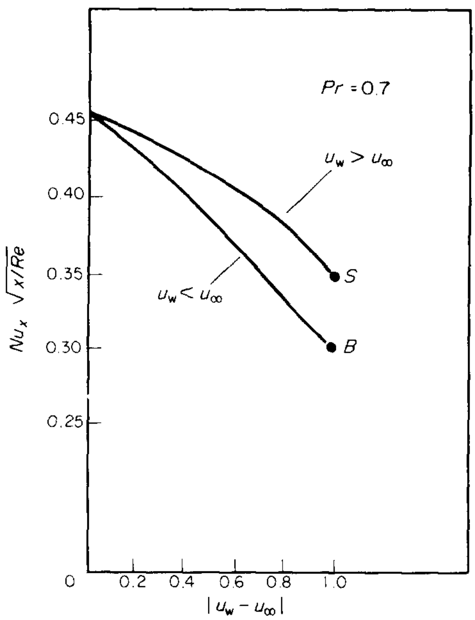

FIG. 3. Heat-transfer coefficient.

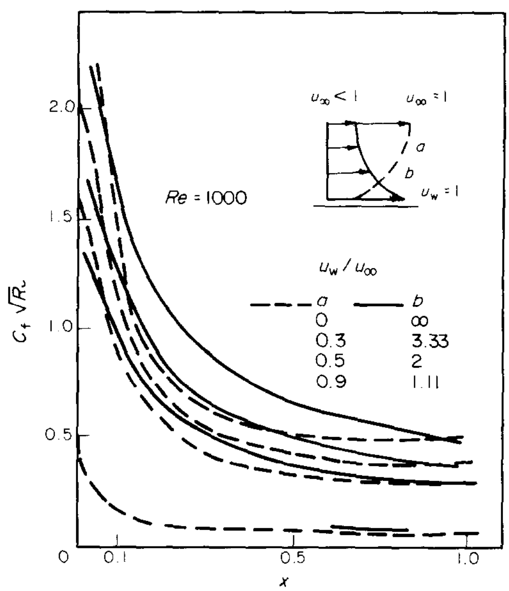

FIG. 5. Navier–Stokes solutions.

than the boundary-layer solution except near the leading edge.

Figure 5 shows Navier–Stokes solutions for the case when the free stream moves faster (case a) and slower (case b) than the flat plate.

5. CONCLUSIONS

For the same Reynolds and Prandtl numbers and the same velocity difference $|u_w - u_\infty|$, larger skin-friction and heat-transfer coefficients result for $u_w > u_\infty$ than for $u_w < u_\infty$. The Navier–Stokes solutions, Fig. 5, qualitatively confirm the previous statement obtained from the boundary-layer theory.

6. REFERENCES

1. H. Blasius, Grenzschichten in Flüssigkeiten mit kleiner Reibung, 2 *Math. Phys.* **56**, 1–37 (1908).
2. B. C. Sakiadis, Boundary-layer behavior on continuous solid surfaces, *A.I.Ch.E. Jl* **7**, 26–28 (1969); **7**, 221–225 (1969).
3. F. Tsou, E. Sparrow and R. Goldstein, Flow and heat transfer in the boundary layer on a continuous moving surface, *Int. J. Heat Mass Transfer* **10**, 219–235 (1967).
4. T. A. Abdelhafez, Die laminare Umströmung einer kontinuierlich pumpenden Platte, Differenzenverfahren zur Integration der Navier-Stokes-Gleichungen.

Dissertation, Fachbereich Maschinentechnik, Universität Essen, B.R.D. (1982).
5. Y. Adam, Highly accurate compact implicit methods and boundary conditions, *J. comput. Phys.* **24**, 10–22 (1977).

7. APPENDIX

We define a new dimensionless, similarity coordinate η as:

$$\eta = y(|u_w - u_\infty)|Re/x)^{0.5}, \quad (21)$$

and a transformed stream function $F(\eta)$ given by:

$$(u_w - u)/(u_w - u_\infty) = F'(\eta). \quad (22)$$

This gives the stream function:

$$\psi(x, y) = \pm(x|u_w - u_\infty|/Re)^{0.5}F(\eta) + u_w y. \quad (23)$$

The upper (positive) sign in the previous as well as the following equations refers to the case $u_\infty > u_w$ and the lower (negative) sign for $u_w > u_\infty$.

If $u_\infty > u_w$, then: $\bar{u}_r = \bar{u}_\infty$, $u_\infty = 1$, $0 < u_w < 1$.

If $u_w > u_\infty$, then: $\bar{u}_r = \bar{u}_w$, $u_w = 1$, $0 < u_\infty < 1$.

It is to be noted that equations (21)–(23) reduce to the similarity transformation of Blasius [1] when $u_w = 0$. The boundary-layer equations may be reduced to:

$$\eta(u_w/|u_\infty - u_w|)F'' \pm FF'' + 2F''' = 0, \quad (24a,b)$$

with the boundary conditions:

$$F'(0) = 0, \quad F'(\eta_\delta) = 1, \quad F(0) = 0. \quad (25)$$

Thus the boundary-layer problem has been reduced to two different ordinary differential equations 24(a,b) to be solved with the boundary conditions (25) as those of Blasius.

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A Stefan problem for exothermic non-catalytic reactions

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NOMENCLATURE

C	concentration
C_p	heat capacity
E	activation energy
H	enthalpy
k	rate constant
L	latent heat
(ΔQ)	heat of reaction
R	gas constant
R_S	rate of reaction
T	temperature
T_*	reference temperature
t	time
z	distance.

Greek symbols

ε	porosity
λ	thermal conductivity
ρ	density.

Subscripts

A, S	solids.

1. INTRODUCTION

HEAT conduction problems involving chemical reaction and phase change are encountered in many areas of interest [1]. Solid–solid exothermic reactions accompanied by melting phenomena are important in pyrotechnics and synthesis of refractory materials. These problems are usually solved by numerical methods [2]. The present communication aims at solving a phase change problem associated with exothermic condensed-phase reacting systems in which the reaction temperature can be sufficiently higher than the melting point of the solid reactant. Thus, there exists a sharply defined phase interface whose rate of movement depends on the rate of heat release, rate of heat conduction and the latent heat of phase change. The problem closely resembles the classical Stefan problem, the difference being the added complication of a non-linear source term.

The enthalpy method is used to solve the system of equations. It has the advantage of not requiring the explicit tracking of the phase change interface. Also, there is no need to consider the solid and liquid regions separately [1, 2].

2. THE PHYSICAL PROBLEM

A heterogeneous reaction occurring between a solid S and solid A is considered:

$$A(s) + S(s) \rightarrow P(s). \quad (1)$$

The reaction rate may be represented by the following equation

$$-R_S = kC_S. \quad (2)$$

The temperature dependence of the reaction rate constant is of Arrhenius type.

The heterogeneous system consisting of solid particles may be treated as though it were homogeneous and a hypothetical continuum is considered.

All system physical properties are constant.

The problems associated with mass diffusion of the solid phase and the phase equilibrium are not considered.

With these assumptions the governing mass balance and

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